

QUANTUM-FIELD THEORIES AS REPRESENTATIONS OF A SINGLE *-ALGEBRA

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ABSTRACT. We show that many well-known quantum field theories emerge as representations of a single $*$ -algebra. These include free quantum field theories in flat and curved space-times, lattice quantum field theories, Wightman quantum field theories, and string theories. We prove that such theories can be approximated on lattices, and we give a rigorous definition of the continuum limit of lattice quantum field theories.

1. INTRODUCTION

The Wightman distributions play a fundamental role in Wightman quantum field theories (Wightman QFTs) [1]. The reconstruction theorem demonstrates that knowledge of the Wightman distributions is sufficient to obtain a unique Wightman QFT. In particular, the Wightman distributions define a state of a Borchers-Uhlmann (BU) algebra, and the associated Wightman QFT is obtained as a representation of a BU-algebra from that state [2, 3]. Interestingly, interacting as well as non-interacting Wightman QFTs emerge as representations of the same BU algebra.

Moreover, in Ref. [4] a similar route is chosen to define QFTs in curved space-times from an axiomatic point of view. Starting point is a free $*$ -algebra, $\text{Free}(M)$, of quantum fields on a background structure, M , which, amongst others, refers to a globally hyperbolic space-time. The quantum-field algebra, $A(M)$, is obtained by factoring $\text{Free}(M)$ by a set of relations, in which coefficients of an operator-product expansion (OPE) play a fundamental role, i.e. there exists a $*$ -homomorphism, $\pi : \text{Free}(M) \rightarrow A(M)$, which essentially is defined by properties of the OPE coefficients. The set of states of the theory, $S(M)$, is further constrained to support the OPE, positivity, and a certain spectrum condition. However, if we construct a representation of $A(M)$ from a state $\omega \in S(M)$, then we basically obtain a representation of $\text{Free}(M)$ with respect to the state $\omega \circ \pi$. The set of states of the QFT can therefore also be seen as a subset of the set of states over $\text{Free}(M)$, which is obtained by appropriate constraints. Quite different QFTs can therefore emerge as representations of the same $*$ -algebra, $\text{Free}(M)$. We pick up this idea and explore it in a more general approach in this paper.

In Sec. 2.1, we introduce the general setting, and we give definitions of the terms test-function space, quantum field, and QFT. The definitions are purely practically motivated

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in order to make the general approach suitable for the various QFTs that we discuss in subsequent sections. In particular, we refrain from discussing any physical implications. However, in Sec. 2.2 we define in which sense a QFT emerges from a representation of a $*$ -algebra (i.e. the polynomial algebra generated by a quantum field), and in Sec. 2.3 we introduce a quantum field whose polynomial algebra can be represented in any QFT, so that the QFT actually emerges from that representation. We also prove that such QFTs can be approximated on lattices, and we give a rigorous definition of the continuum limit. These are the central results of this paper.

We discuss in Sec. 3 the applicability of the approach to hermitian scalar Wightman QFTs, and in Sec. 4 we discuss free scalar QFTs and Dirac QFTs in curved space-times. In Sec. 5, we continue with the discussion of perturbatively interacting quantum fields in curved space-times, and we consider lattice QFTs and string theories in Sec. 6. In particular, we discuss the continuum limit of lattice QFTs more concretely.

2. ABSTRACT QUANTUM FIELDS

2.1. General setting. Let us begin with the definition of some general terms. We note however that we do not aim at comprehensive, generally accepted definitions in this paper, but that we rather restrict ourselves to working definitions that suit the mathematical requirements and that fit to the QFTs that we discuss in subsequent sections.

In Wightman QFTs, quantum fields are defined as operator-valued distributions over a test-function space. We choose a similar approach:

- (1) A conjugation, C , on a complex vector space, V , is an antilinear map satisfying $C^2 = 1$ and $C(av + bw) = \bar{a}C(v) + \bar{b}C(w)$ for all $a, b \in \mathbb{C}$ and all $v, w \in V$.
- (2) Let V_1 and V_2 be vector spaces with conjugations C_1 and C_2 . A c -homomorphism, $h : V_1 \rightarrow V_2$, is a vector-space homomorphism that is compatible with the conjugations, i.e. $h \circ C_1 = C_2 \circ h$.
- (3) A test-function space is a complex separable locally-convex Hausdorff topological vector space on which a conjugation is defined.
- (4) A quantum field is a complex-linear map from a test-function space into a $*$ -algebra, $\Phi : V \rightarrow P(\Phi)$, where $P(\Phi)$ is the polynomial algebra generated by the set of operators $\Phi(V) \cup \{1\}$.¹ Φ further satisfies $\Phi(f)^* = \Phi(Cf)$ for all $f \in V$.
- (5) A quantum-field theory is a pair (Φ, ω) , where Φ is a quantum field, ω is a state over $P(\Phi)$, and, for all $n \in \mathbb{N}$,

$$\omega \left(\prod_{m=1}^n \Phi(\cdot) \right)$$

is a multi-linear continuous functional on V^n .

Let us elaborate on our definition. An example of a test-function space is Schwartz space, $S(\mathbb{R}^n)$. Complex conjugation is given in $S(\mathbb{R}^n)$ by

$$C(af + bg)(x) = \bar{a}\bar{f}(x) + \bar{b}\bar{g}(x) = \bar{a}(Cf)(x) + \bar{b}(Cg)(x).$$

¹We note that if the field operators satisfy CCRs or CARs, then $P(\Phi)$ is generated by $\Phi(V)$ alone.

Moreover, for any locally-convex topological vector space, V , a test-function space can be constructed as follows. Let \bar{V} denote the corresponding complex conjugate vector space (c.f. appendix A.2 in Ref. [5]), and let $j : V \rightarrow \bar{V}$ denote the natural anti-linear bijection. The set $\{j(B) : B \text{ open in } V\}$ is a locally-convex Hausdorff topology on \bar{V} . Let $W = V \oplus \bar{V}$ be equipped with the product topology of $V \times \bar{V}$, and define the conjugation $C(f, g) = (j^{-1}(g), j(f))$, then W is a test-function space.

A test-function space, V , always has an associated field, which can be constructed as follows. Let

$$\mathcal{A}_V = \bigoplus_{n=0}^{\infty} V^{\otimes n}$$

denote the tensor algebra of V . For the sake of notational convenience, we denote an element in $V^{\otimes n}$ by

$$v_1 \otimes v_2 \otimes \dots \otimes v_n = \bigotimes_{m=1}^n v_m$$

Moreover, we define the involution

$$\begin{aligned} C^{(n)} \left(\bigotimes_{m=1}^n v_m \right) &= \bigotimes_{m=1}^m C(v_{n-m+1}) \quad (v_m \in V) \\ (a_n)^* &= (C^{(n)} a_n) \quad ((a_n) \in \mathcal{A}_V), \end{aligned}$$

so that \mathcal{A}_V is a (non-commutative) *-algebra. We further define the complex-linear quantum field, $\Phi_V : V \rightarrow \mathcal{A}_V$, by

$$\Phi_V(v) = (0, v, 0, 0, \dots) \quad (v \in V),$$

then $P(\Phi_V) = \mathcal{A}_V$. Note that for $V = S(\mathbb{R}^n)$, \mathcal{A}_V is a BU-algebra.

Multiple quantum fields can be combined into a single quantum field as follows. Let $\Phi_i : V_i \rightarrow P(\Phi_i)$ ($i \in I$) be at most countably many quantum fields, for which multiplication and addition of field operators are defined. Define

$$V = \bigoplus_i V_i, \quad C(v_i) = (C_i v_i),$$

and equip V with the product topology, i.e. V is a subspace of $\prod_i V_i$. V is a complex separable locally-convex Hausdorff topological vector space, i.e. V is a test-function space. Define further the map

$$\Phi((f_i)) = \sum_i \Phi_i(f_i), \quad \Phi((f_i))^* = \sum_i \Phi_i(f_i)^*,$$

then Φ is a quantum field.

Proposition 1: Let $\Phi : V \rightarrow P(\Phi)$ be a quantum field, let W be a test-function space, and let $h : W \rightarrow V$ be a c-homomorphism, then $\Phi \circ h$ induces a *-homomorphism, $\pi : P(\Phi_W) \rightarrow P(\Phi)$, so that $\Phi \circ h = \pi \circ \Phi_W$.

Proof: $\Phi \circ h$ is a complex-linear function from W into $P(\Phi)$ that uniquely extends to an

algebra homomorphism, π , from the tensor algebra \mathcal{A}_W of W to $P(\Phi)$ by the universal property of the tensor algebra. Let C_V denote the conjugation on V , and let C_W denote the conjugation on W . π further is a *-homomorphism:

$$\pi((a_n)) = a_0 + \sum_n \bigotimes_{m=1}^n \Phi(h(w_{m,n})) \quad ((a_n) \in \mathcal{A}_W, a_n = \bigotimes_{m=1}^n w_{m,n} \in W^{\otimes n}, w_{m,n} \in W),$$

and

$$\begin{aligned} \pi((a_n))^* &= \bar{a}_0 + \sum_n \bigotimes_{m=1}^n \Phi(h(w_{n-m+1,n}))^* = \bar{a}_0 + \sum_n \bigotimes_{m=1}^n \Phi(C_V(h(w_{n-m+1,n}))) = \\ &= \bar{a}_0 + \sum_n \bigotimes_{m=1}^n \Phi(h(C_W w_{n-m+1,n})) = \pi((a_n)^*). \end{aligned}$$

In particular, $(\Phi \circ h)(w)^* = (\pi \circ \Phi_W)(w)^* \quad (w \in W)$. ■

Definition 1: Let $\Phi_i : V_i \rightarrow P(\Phi_i)$ be quantum fields ($i = 1, 2$). Φ_1 is a core for Φ_2 if there exists a continuous c-homomorphism, $h : V_1 \rightarrow V_2$, and a *-homomorphism, $\pi : P(\Phi_1) \rightarrow P(\Phi_2)$, so that $h(V_1)$ is dense in V_2 and that $\Phi_2 \circ h = \pi \circ \Phi_1$. If h is surjective, then Φ_2 is a quotient of Φ_1 .

Corollary 1: Let $\Phi : V \rightarrow P(\Phi)$ be a quantum field, then Φ is a quotient of Φ_V .

Proof: The identity map, $\text{id} : V \rightarrow V$, is a surjective continuous c-homomorphism, so that proposition 1 yields $\Phi = \pi \circ \Phi_V$. ■

Let us assume $\Phi_2 \circ h = \pi \circ \Phi_1$ as in definition 1, i.e. Φ_1 is a core for Φ_2 , and let (Φ_2, ω_2) be a QFT. $\omega_1 = \omega_2 \circ \pi$ is a state on $P(\Phi_1)$, and (Φ_1, ω_1) is also a QFT. We argue in the following that (Φ_1, ω_1) and (Φ_2, ω_2) essentially yield the same quantum theory.

2.2. Quantum theories from quantum-field cores. Let (Φ, ω) be a QFT. ω induces the representation of $P(\Phi)$ on a pre-Hilbert space, D_ω , by the GNS construction (for *-algebras). Elements in D_ω are given by equivalence classes of operators in $P(\Phi)$. The representation is commonly denoted by $(H_\omega, \pi_\omega, \Omega_\omega)$, where H_ω is the completion of D_ω , π_ω is a *-homomorphism, and Ω_ω is the unit vector corresponding to the unity operator, 1. Expectation values of operators in $P(\Phi)$ are given by

$$\langle u_{[a]}, \pi(b)u_{[c]} \rangle = \langle u_{[a]}, u_{[bc]} \rangle = \omega(a^*bc) \quad (a, b, c \in P(\Phi)).$$

The continuity property in our definition of a QFT further guarantees that

$$\langle u, \prod_{i=1}^n \Phi(f_i)v \rangle$$

defines a complex-linear multi-continuous functional for all $u, v \in D_\omega$. This is satisfied in scalar Wightman QFTs, for example.

However, let τ_ω denote the locally-convex topology generated by the set of semi-norms, $\{\|\pi_\omega(\cdot)u\| : u \in D_\omega\}$, on $P(\Phi)$. Let us further assume that V contains a subset, V_s , so that $\pi_\omega(\Phi(f))$ is essentially self-adjoint for all $f \in V_s$ and that $P(\Phi)$ is generated by $\Phi(V_s)$. We

call such a QFT regular. Let \mathcal{A} be the C^* -algebra generated by the set $\{\exp(i\pi_\omega(\Phi(f))) : f \in V_s\}$, then $\pi_\omega(P(\Phi))' = \mathcal{A}'$, i.e. the commutants agree, and $\pi_\omega(P(\Phi))'' = \mathcal{A}''$. \mathcal{A}'' may be seen as the algebra of observables of the state ω , which contains all projection-valued measures that are relevant in the respective quantum theory (c.f. Def. 2.6.3 in Ref. [6]). If the set $\{\Phi(f) : f \in V_s\}$ is irreducible, then the set $\{\exp(i\pi_\omega(\Phi(f))) : f \in V_s\}$ is also irreducible, and \mathcal{A}'' equals the set of linear-bounded operators on H_ω . However, by von-Neumann's density theorem, \mathcal{A} is dense in \mathcal{A}'' with respect to the weak operator topology, and the restriction of any operator in \mathcal{A}'' to D_ω is the τ_ω -limit of a net of polynomials of field operators.

Let $\tilde{\Phi}$ be a core for Φ , i.e. there exists a continuous c -homomorphism, h , and a $*$ -homomorphism, π , so that $\Phi \circ h = \pi \circ \tilde{\Phi}$, then $\pi(P(\tilde{\Phi}))$ is dense in $P(\Phi)$ with respect to the τ_ω -topology. In particular, each operator in $\pi_\omega(P(\Phi))$ is the strong-graph limit of a net of operators in $\pi_\omega \circ \pi(P(\tilde{\Phi}))$. If (Φ, ω) is regular, then $\pi_\omega \circ \pi(P(\tilde{\Phi}))'' = \mathcal{A}''$.

Proposition 2: Let (Φ, ω) be a QFT, and let $\tilde{\Phi}$ be a core for Φ , then there exists a $*$ -homomorphism, π , so that each operator in $\pi_\omega(P(\Phi))$ is the strong-graph limit of a net of operators in $\pi_\omega \circ \pi(P(\tilde{\Phi}))$.

Let us re-formulate proposition 2 into a looser statement: (Φ, ω) and $(\tilde{\Phi}, \omega \circ \pi)$ essentially yield the same quantum theory, which emerges from a representation of $P(\tilde{\Phi})$.

2.3. A universal quantum field. Let V_0 be the vector space of cofinite complex sequences,

$$V_0 = \bigoplus_{n=1}^{\infty} \mathbb{C},$$

and define the conjugation

$$C_0((c_n)) = (\bar{c}_n) \quad ((c_n) \in V_0).$$

Lemma 1: Let V be a test-function space. There exists a c -homomorphism, $h : V_0 \rightarrow V$, so that $h(V_0)$ is dense in V . If V is finite-dimensional, then h is surjective.

Proof: Let $\{f_n\}$ be a countable, dense subset of V , let W be the linear span of $\{f_n\}$, and let C denote the conjugation on V . Let further $\{e_n\}$ be a maximal linear-independent subset of

$$\{(f_n + C(f_n)) \cup i(f_n - C(f_n))\},$$

then $\{e_n\}$ is a basis of W . Note that if W is finite-dimensional, then W is closed and thus $W = V$. Moreover, the function

$$h((c_n)) = \sum_{n=1}^{\dim W} c_n e_n \quad ((c_n) \in V_0)$$

is a vector-space homomorphism between V_0 and W , which is compatible with the conjugations,

$$C(h((c_n))) = \sum_n \bar{c}_n e_n = h(C_0((c_n))). \quad \blacksquare$$

Let \mathcal{V} be the class of test-function spaces, and for $V \in \mathcal{V}$ let $h_V : V_0 \rightarrow V$ be a c -homomorphism so that $h(V_0)$ is dense in V . If $V = V_0$ as sets, and if the conjugation on V is C_0 , then we choose $h_V = \text{id}$. We equip V_0 with the initial topology generated by the set $\{h_V^{-1}(B) : B \text{ open in } V, V \in \mathcal{V}\}$. Note that this is the weakest topology for which all functions h_V ($V \in \mathcal{V}$) are continuous.

Lemma 2: V_0 is a test-function space.

Proof: We first note that a function $f : Z \rightarrow V_0$ is continuous if and only if $h_V \circ f$ is continuous for all $V \in \mathcal{V}$. Let $f_{c,W}$ denote multiplication with $c \in \mathbb{C}$ on a vector space, W . f_{c,V_0} is continuous since $h_V \circ f_{c,V_0} = f_{c,V} \circ h_V$, and since scalar multiplication is continuous on V for all $V \in \mathcal{V}$. Let $g_W(a, b) = a + b$ denote the addition function on a vector space, W . g_{V_0} is continuous since $(h_V \circ g_{V_0})(a, b) = g_V(h_V(a), h_V(b)) = h_V(a) + h_V(b)$, and since addition is continuous on V for all $V \in \mathcal{V}$. Let further $a, b \in V_0$, $a \neq b$, let $V = V_0$ as sets, let V be equipped with the topology induced by the semi-norms $p_m((c_n)) = |c_m|$ ($(c_n) \in V$), and let C_0 be the conjugation on V , then V is a test-function space. The topology of V is contained in the topology of V_0 by definition of V_0 . Since V is Hausdorff, there exist open sets A and B in V so that $a \in A$, $b \in B$, and $A \cap B = \emptyset$. V_0 is therefore a Hausdorff space since A and B are also open in V_0 . For each $V \in \mathcal{V}$ let \mathcal{B}_V denote a neighborhood base of 0 of balanced, convex, absorbing sets, and let \mathcal{B}_{V_0} be the set of finite intersections of sets in $\{h_V^{-1}(B) : B \in \mathcal{B}_V, V \in \mathcal{V}\}$. \mathcal{B}_{V_0} is a neighborhood base of 0. Due to linearity, $h_V^{-1}(B)$ is a balanced, convex, and absorbing set for all $B \in \mathcal{B}_V$ and all $V \in \mathcal{V}$. Let $B \in \mathcal{B}_{V_0}$, $B = C_1 \cap \dots \cap C_n$, and let $a \in V_0$. B is balanced and convex. For each C_i ($1 \leq i \leq n$) there exists a $t_i > 0$ so that $a \in tC_i$ if $t \geq t_i$. Let $t_0 = \max\{t_1, \dots, t_n\}$, then $a \in tB$ if $t \geq t_0$. B is therefore absorbing, and V_0 is locally convex. Moreover, V_0 is the union of countably many finite-dimensional spaces,

$$V_0 = \bigcup_n V_{0,n}, \quad V_{0,n} = \{(c_m) \in V_0 : c_m = 0 \forall m > n\}.$$

Since each $V_{0,n}$ is finite-dimensional, the respective subspace topologies are equivalent to the Euclidian topologies, which entails that each $V_{0,n}$ is separable. Let $W_{0,n}$ be a countable, dense subset of $V_{0,n}$, then $\bigcup_n W_{0,n}$ is a countable, dense subset of V_0 . ■

Let $\Phi_0 : V_0 \rightarrow \mathcal{A}_0$ be the quantum field associated with V_0 . Lemma 1 and proposition 1 yield the following theorem.

Theorem 1: Let $\Phi : V \rightarrow P(\Phi)$ be a quantum field, then Φ_0 is a core for Φ . If V is finite-dimensional, then Φ is a quotient of Φ_0 .

Considering proposition 2, Φ_0 actually is a universal quantum field, since any QFT basically is a QFT of Φ_0 , which emerges from a representation of $P(\Phi_0)$.

Theorem 2: Let ω be a state over $P(\Phi_0)$, then (Φ_0, ω) is a QFT.

Proof: We need to show that, for all $n \in \mathbb{N}$,

$$\omega^{(n)} = \omega \left(\prod_{m=1}^n \Phi_0(\cdot) \right)$$

is a multi-linear continuous functional on V_0^n . Let F be the set of linear functions from V_0 to \mathbb{C} . For each $f \in F$ we define the semi-norm $p_f(v) = |f(v)|$ ($v \in V_0$). The set of semi-norms, $\{p_f\}_{f \in F}$, defines a locally-convex Hausdorff topology on the set V_0 . Let V_0^F denote the corresponding topological space. We choose C_0 as conjugation on V_0^F , so that V_0^F is a test-function space. Each $f \in F$ is continuous when considered as a function from V_0^F to \mathbb{C} . Due to the definition of the test-function space V_0 , each open set in V_0^F is also open in V_0 , so that each $f \in F$ is also continuous when considered as a function from the test-function space V_0 to \mathbb{C} . Hence, $\omega^{(n)}$ is continuous in each argument, and therefore it is continuous on V_0^n . ■

Let S_0 be the set of states over $P(\Phi_0)$. Each $a \in P(\Phi_0)$ defines a linear functional on S_0 by $l_a(\omega) = \omega(a)$ for $\omega \in S_0$. The corresponding set of semi-norms, $p_a(\omega) = |l_a(\omega)| = |\omega(a)|$, defines a topology on S_0 , and a net (ω_i) in S_0 converges to an $\omega \in S_0$ with respect to that topology, if $\lim_i \omega_i(a) = \omega(a)$ for all $a \in P(\Phi_0)$.

Let $V_{0,n} = \{(c_m) \in V_0 : c_m = 0 \forall m > n\}$. We will argue in Sec. 6.1 that $V_{0,n}$ is the test-function space of a quantum system with n degrees of freedom. Such quantum systems typically are considered in lattice QFTs. However, let $\Phi_{0,n}$ denote the corresponding quantum field, and let $P_n : V_0 \rightarrow V_{0,n}$ be the canonical projection, i.e. $P_n((c_m)) = (c_1, \dots, c_n, 0, 0, \dots)$ for $(c_m) \in V_0$, then $\Phi_{0,n} = \Phi_0 \circ P_n$ and $P(\Phi_{0,n})$ is a sub-algebra of $P(\Phi_0)$. Since P_n is a \mathbb{C} -homomorphism, there exists a corresponding $*$ -homomorphism, $\pi_n : P(\Phi_0) \rightarrow P(\Phi_{0,n})$, by proposition 1 so that $\Phi_{0,n} = \pi_n \circ \Phi_0$. Let (Φ_0, ω) be a QFT, then $\omega_{(n)} = \omega|_{P(\Phi_{0,n})}$ defines a state on $P(\Phi_{0,n})$, and $(\Phi_0, \omega_{(n)} \circ \pi_n)$ is a QFT. For the sake of notational convenience we denote $\omega_{(n)} \circ \pi_n$ simply by $\omega_{(n)}$ in the following. $(\Phi_0, \omega_{(n)})$ represents a reduced quantum system. We note that the series $(\omega_{(n)})$ converges to ω , and that the QFTs discussed in this paper can therefore be approximated on lattices, see also Sec. 6.1.

Theorem 3: Let (ω_i) be a net of states over $P(\Phi_0)$. (ω_i) converges to a state ω if and only if $(\omega_{i,(n)})$ converges to $\omega_{(n)}$ for all $n \in \mathbb{N}$.

Proof: $\omega_{i,(n)} \rightarrow \omega_{(n)}$ is a consequence of $\omega_i \rightarrow \omega$. Let us assume that $\omega_{i,(n)} \rightarrow \omega_{(n)}$ for all $n \in \mathbb{N}$, and let $a \in P(\Phi_0)$. Since $a \in P(\Phi_{0,n})$ for some $n \in \mathbb{N}$, and since for $n < m$, $V_{0,n} \subset V_{0,m}$ and $\omega_{(n)} = \omega_{(m)}|_{P(\Phi_{0,n})}$, the states $\omega_{(n)}$ defines a unique state ω on $P(\Phi_0)$. ■

In the remaining part of this paper, we will show that theorem 1 applies to many well-known QFTs, and that theorems 2 and 3 define the continuum limit of lattice QFTs in our approach.

3. APPLICATION TO WIGHTMAN QUANTUM FIELD THEORIES

The relation of our approach to Wightman QFTs can be conveniently discussed with the help of the Wightman reconstruction theorem. For the sake of convenience, let us consider a hermitian scalar Wightman QFT. In the reconstruction theorem, Wightman QFTs are recovered as representations of a Borchers-Uhlmann algebra [1]. The test-function space in Wightman QFTs is Schwartz space, $S(\mathbb{R}^d)$ ($d \geq 2$). Using the terminology and the definitions of Sec. 2, the Borchers-Uhlmann algebra is given by \mathcal{A}_V with $V = S(\mathbb{R}^d)$, and the corresponding quantum field is denoted by $\Phi_V : V \rightarrow \mathcal{A}_V$. Note that $S(\mathbb{R}^d)$ is separable, and that corollary 1 and theorem 1 apply.

Corollary 2: Let (Φ, ω) be a hermitian scalar Wightman QFT, then Φ is a quotient of Φ_V ($V = S(\mathbb{R}^d)$), and Φ_0 is a core for Φ .

We note that corollary 2 applies to any (hermitian scalar) Wightman QFT involving $d \geq 2$ space-time dimensions, and to Wightman QFTs of interacting quantum fields as well as free quantum fields. Moreover, considering proposition 2, we can re-formulate corollary 2 into a looser statement: There exists a $*$ -homomorphism, π , so that (Φ, ω) and $(\Phi_0, \omega \circ \pi)$ essentially yield the same quantum theory, which emerges from a representation of $P(\Phi_0)$.

Let us discuss two examples of Wightman QFTs in more detail. In general, the set of field operators in Wightman QFTs is irreducible. Moreover, let V_s denote the subset of real functions in $S(\mathbb{R}^d)$, then the set $\{\Phi(f) : f \in V_s\}$ is also irreducible, and $\Phi(V_s)$ generates $P(\Phi)$. In the usual Fock-space representation of free scalar fields [7], the operators $\Phi(f)$ ($f \in V_s$) are essentially self-adjoint, i.e. the QFT is regular. Unfortunately the situation is less straightforward for Wightman QFTs of interacting fields, since there do not exist that many examples. However, let us consider $P(\varphi)_2$ as presented in Ref. [8].

$P(\varphi)_2$ is defined in flat space-time with one time dimension and one space dimension. The corresponding Fock space, \mathcal{F} , of the free hermitian scalar QFT is the symmetric tensor algebra over $L_2(\mathbb{R})$. For each open bounded interval, $B \subset \mathbb{R}$, let $\mathcal{A}(B)$ denote the von-Neumann algebra generated by the operators $\exp(i\varphi(f_1) + i\pi(f_2))$ ($f_1, f_2 \in C_0^\infty(B)$, f_1, f_2 real) and let \mathcal{A} denote the norm closure of $\bigcup_B \mathcal{A}(B)$. $P(\varphi)_2$ is constructed by considering the GNS representation, $(H_\omega, \pi_\omega, \Omega_\omega)$, of \mathcal{A} with respect to a specific state, ω . In this representation, the unitary groups

$$W_t(f_1, f_2) = \pi_\omega(\exp(it\varphi(f_1) + it\pi(f_2))) \quad (f_1, f_2 \in C_0^\infty(B))$$

are strongly continuous, and they have self-adjoint generators.

Let $\Phi(f) = \varphi(\text{Re}(f)) + i\pi(\text{Im}(f))$, then Φ is a quantum field. Let Φ' be the restriction of Φ to $C_0^\infty(\mathbb{R})$ (assuming the Schwartz-space topology), then Φ' is a core for Φ , and (Φ', ω) is regular. Also, let $W_s = V_s \cap C_0^\infty(\mathbb{R})$, then the set $\{\Phi(f) : f \in W_s\}$ is irreducible, i.e. $\mathcal{A} \subset P(\Phi')'' = B(H_\omega)$. We can therefore say that (Φ, ω) and (Φ', ω) essentially yield the same quantum theories. Moreover, Φ_0 is a core for Φ' , and proposition 2 applies.

4. APPLICATION TO FREE QUANTUM FIELD THEORIES

In this section, we discuss the relation of our general approach to free QFTs that implement canonical commutation relations (CCRs) or canonical anti-commutation relations (CARs) on Fock space. We first discuss both cases together without specifying if the Fock space, \mathcal{F} , is symmetric or anti-symmetric. We assume however that \mathcal{F} is constructed over an infinite-dimensional, separable Hilbert space, \mathcal{H} .

In conventional representations [9], the annihilation and creation operators, $a(f)$ and $a^*(f)$, are both defined over \mathcal{H} , so that $a(f)$ is complex anti-linear and that $a^*(f)$ is complex linear. We note that $a(f)$ and $a^*(f)$ are densely defined, closed, and that $a(f)^* = a^*(f)$. If there is a complex conjugation, C , defined on \mathcal{H} , then \mathcal{H} and \mathcal{H}^2 are test-function spaces, and we can introduce the complex-linear field $\Phi_1(f, g) = a^*(f) + a(Cg)$. Since $\Phi_1(f, 0) = a^*(f)$ and $\Phi_1(0, Cf) = a(f)$, $P(\Phi_1)$ is the polynomial algebra generated by the irreducible set of operators $A = \{a(f), a^*(f)\}_{f \in \mathcal{H}}$. In particular, in the symmetric case (CCRs), we obtain

$$\Phi(f) = \frac{a^*(f) + a(f)}{\sqrt{2}} = \frac{\Phi_1(f, 0) + \Phi_1(0, Cf)}{\sqrt{2}}.$$

Since \mathcal{H}^2 is separable, we can apply theorem 1.

Corollary 3: Φ_0 is a core for Φ_1 .

The annihilation operators in free QFTs can conveniently be chosen as complex-linear operator-valued functionals if they are defined over the dual Hilbert space instead, i.e. one considers $a(f)$ with $f \in \mathcal{H}^*$ [5]. One advantage of this choice is that vacuum expectation values become multi-linear functionals over the test-function space. However, by the Riesz lemma, there is a natural complex anti-linear bijection, $j : \mathcal{H} \rightarrow \mathcal{H}^*$, and we can define the corresponding complex conjugation on $W = \mathcal{H} \otimes \mathcal{H}^*$ by $C(f, g) = (j^{-1}(g), j(f))$, i.e. W is a test-function space (c.f. Sec. 2). In particular, let $W_2 = l_2(\mathbb{N}) \otimes l_2(\mathbb{N})^*$, and let $\Phi_2 = \Phi_{W_2}$. For a separable Hilbert space, \mathcal{H} , choose a unitary operator $U : l_2(\mathbb{N}) \rightarrow \mathcal{H}$, let $\bar{U} : \mathcal{H}^* \rightarrow l_2(\mathbb{N})^*$ denote the corresponding dual unitary operator, define $\Phi'_2(f, g) = a^*(f) + a(g)$, and define $\pi_U(\Phi_2(f, g)) = \Phi'_2(Uf, \bar{U}g)$. π_U is a *-homomorphism between $P(\Phi_2)$ and $P(\Phi'_2)$, Φ'_2 is a quotient of Φ_2 , and the Fock representation is a *-homomorphic representation of $P(\Phi_2)$. We summarize.

Corollary 4: In any free QFT, the polynomial algebra generated by the irreducible set of operators $\{a(f), a^*(g)\}_{f \in \mathcal{H}, g \in \mathcal{H}^*}$ is *-homomorphic to $P(\Phi_2)$, and Φ_0 is a core for Φ_2 .

Adopting the same loose language as after proposition 2, we state that free QFTs emerge from Fock representations of $P(\Phi_0)$.

Let us discuss two examples of free QFTs. In Ref. [10], CCRs are defined with $\mathcal{H} = L_2(S)$, where S is a Cauchy surface of a globally hyperbolic manifold. Independence of the actual choice of S is due to a *-isomorphism between representations of the CCRs over the same vector space ($C_0^\infty(M)$). We note that a more detailed account is given in Ref. [5], where especially the arbitrariness of the choice of scalar product in the definition of

\mathcal{H} is discussed. However, we note that corollary 3 applies irrespective of the specifically chosen globally hyperbolic manifold and background metric, i.e. the QFTs emerge from representations of $P(\Phi_0)$ and $P(\Phi_2)$, respectively.

The second example are Dirac quantum fields on a globally hyperbolic manifold, M , that are constructed in Ref. [11]. The construction is based on the definition of a scalar product on $C_0^\infty(DS)$, where DS is the Dirac spinor bundle of spinors on a Cauchy surface, S . DS is a vector bundle, and $C_0^\infty(DS)$ is locally isomorphic to $C_0^\infty(S)^4$. The completion of $C_0^\infty(DS)$ yields a separable Hilbert space, \mathcal{H} . The dual Hilbert space, \mathcal{H}^* , is the closure of $C_0^\infty(D^*S)$, the space of cross sections with compact support over the dual vector bundle D^*S . The representation of CARs over S is further defined as a representation of the CARs over the pair $\mathcal{H}, \mathcal{H}^*$. We note that corollary 4 applies irrespective of the specifically chosen globally hyperbolic manifold and background metric, i.e. the QFTs emerge from representations of $P(\Phi_0)$ and $P(\Phi_2)$, respectively.

5. APPLICATION TO PERTURBATIVE QUANTUM FIELD THEORY IN CURVED SPACE-TIMES

The perturbative approach to quantum theories of interacting fields in curved space-time is based on the quantum theory of free fields. In conventional approaches (c.f. Sec. 4), free quantum fields in curved space-time are operator-valued distributions over a space of smooth, compactly-supported test functions. The corresponding quantum-field algebras, i.e. the polynomial algebras generated by the free quantum fields, are however too small to contain the stress-energy operator, for example [5]. The first step therefore is to enlarge the quantum-field algebras with the help of microlocal analysis. Let us consider a QFT of free scalar hermitian fields as it is presented in Ref. [12].

The test function space of a free scalar hermitian QFT on a globally hyperbolic space-time manifold, M , is $C_0^\infty(M)$. Assuming a quasi-free Hadamard state, we can represent the the quantum-field algebra by the GNS construction. In such a representation, however, the quantum fields are operator-valued distributions over a larger, distributional test-function space. Let

$$W_n(x_1, \dots, x_n) =: \phi(x_1) \dots \phi(x_n) :_\omega$$

denote the (Wick-ordered) operator-valued distribution that is defined over $C_0^\infty(M^n)$ and let ω denote a quasi-free Hadamard state ($n \geq 1, W_0 = 1$). As can be shown by microlocal analysis, the operator-valued distributions are defined on a larger space, E'_n , which contains $C_0^\infty(M^n)$ and which is a subspace of the dual space $C_0^\infty(M^n)'$. Distributions in E'_n are compactly supported, and they satisfy the wave-front condition $\text{WF}(t) \subset G_n$ ($t \in E'_n$), where $G_n = (T^*M)^n \setminus H_n$ and

$$H_n = \{(x, k) : x \in M, k \in (\bar{V}^+)^n \cup (\bar{V}^-)^n\}$$

Note that such distributions can be multiplied with each other, so that local Wick polynomials can rigorously be defined. Let \mathcal{W} be the *-algebra of operators generated by 1 and elements $\{W_n(t)\}_{n \in \mathbb{N}, t \in E'_n}$ (c.f definition 2.1 in Ref. [12]). Note also that E'_n does not have a proper topology in the sense of Hörmander [13], but that it rather has a so-called pseudo topology, see Ref. [12] for more details.

However, let us endow E'_n with the sub-space topology, which is inherited from the weak* topology on $C_0^\infty(M^n)'$. E'_n is separable and complex conjugation is well-defined since $C_0^\infty(M^n)$ is a dense subspace, i.e. E'_n is a test-function space according to our definition in Sec. 2. Each W_n is a quantum field, and the corresponding polynomial algebra, $P(W_n)$, is a sub-algebra of \mathcal{W} . As outlined in Sec. 2, we can combine the fields by defining the test-function space $E' = \bigoplus_n E'_n$ and the field $\Phi_3 = \sum_n W_n$, so that we obtain $P(\Phi_3) = \mathcal{W}$. We apply theorem 1.

Corollary 5: Φ_0 is a core for Φ_3 .

In the perturbative approach, interacting fields are formally defined by a perturbation series. Series of operators in \mathcal{W} form an algebra $\mathcal{X} = \mathcal{W}^\mathbb{N}$ with multiplication $(a_n) \star (b_n) = (a_0 b_0, a_1 b_0 + a_0 b_1, \dots)$. Note that the product is defined as if one formally multiplies $\sum_n a_n$ and $\sum_n b_n$. Let us consider the test-function space $\bigoplus_n E'$, on which we define the quantum field $\Phi_4(f) = (\Phi_3(f_n))$. Using the CCRs, one can further show that $P(\Phi_4)$ contains all co-finite sequences in \mathcal{X} . Note that the co-finite sequences are dense in \mathcal{X} if we consider a weak topology on \mathcal{W} and the product topology on \mathcal{X} . However, we can again apply theorem 1.

Corollary 6: Φ_0 is a core for Φ_4 .

Moreover, the interacting-field algebra is defined in Ref. [14] as a sub-algebra of \mathcal{X} as follows. There exists a multi-linear map

$$T_{L_1}^{(n)} : \mathcal{D}_1(M, \mathcal{V})^n \rightarrow \mathcal{X},$$

where $\mathcal{D}_1(M, \mathcal{V})$ is a vector space and L_1 denotes the Lagrangian. \mathcal{V} is a vector space, which is generated by a countably infinite (Hamel) basis, and $\mathcal{D}_1(M, \mathcal{V})$ is the space of compactly-supported smooth densities on M with values in \mathcal{V} . An element $F \in \mathcal{D}_1(M, \mathcal{V})$ can be uniquely expressed as a finite sum, $F = \sum f_i v_i$, where $f_i \in C_0^\infty(M)$ and $v_i \in \mathcal{V}$. As a vector space, $\mathcal{D}_1(M, \mathcal{V})$ is therefore isomorphic to $\bigoplus_n C_0^\infty(M)$. The interacting-field algebra is further defined as the algebra generated by the images of the maps $T_{L_1}^{(n)}$.

Let us endow $\bigoplus_n C_0^\infty(M)$ with the topology induced by $C_0^\infty(M)^\mathbb{N}$, and let the conjugation on $\bigoplus_n C_0^\infty(M)$ be induced by the usual complex conjugation on $C_0^\infty(M)$. $\mathcal{D}_1(M, \mathcal{V})^n$ and $(\bigoplus_m C_0^\infty(M))^n$ are hence test-function spaces for all $n \in \mathbb{N}$, and the maps $T_{L_1}^{(n)}$ satisfy our definition of a quantum field, see Sec. 2. Let Φ_5 denote the quantum field combining the $T_{L_1}^{(n)}$, then the interacting-field algebra is $P(\Phi_5)$. We can again apply theorem 1.

Corollary 7: Φ_0 is a core for Φ_5 .

Moreover, in [15], an algorithm is presented to construct the Wilson operator-product expansion (OPE). The algorithm is generally applicable to perturbative interacting QFT in Lorentzian curved space-times, and it is explicitly presented for the example of a scalar hermitian self-interacting field. However, as proposed in [4], the OPE can actually be elevated to a fundamental level, so that the QFT is determined by its OPE. This yields a general axiomatic framework for QFTs in curved space-times. In particular, the algebra

of interacting quantum fields, \mathcal{F}_i , is obtained by factoring the corresponding free algebra, \mathcal{F}_0 , by a set of relations arising from properties of the OPE coefficients. These relations define an ideal, I , in \mathcal{F}_0 , and $\mathcal{F}_i = \mathcal{F}_0/I$. We note that the corresponding quotient map, $\pi_I : \mathcal{F}_0 \rightarrow \mathcal{F}_i$ is a $*$ homomorphism. As pointed out in this paper so far, we typically can find a $*$ homomorphism, $\pi_0 : P(\Phi_0) \rightarrow \mathcal{F}_0$, so that $\pi_0(P(\Phi_0))$ is dense in \mathcal{F}_0 with respect to an appropriate topology. Then, $\pi_I \circ \pi_0$ is a $*$ homomorphism that maps $P(\Phi_0)$ onto a dense set in \mathcal{F}_i with respect to another topology. Adopting the same loose language as after proposition 2, we then can state that a perturbative interacting QFT emerges from a representation of $P(\Phi_0)$, if the corresponding free QFT does.

6. FURTHER REPRESENTATIONS OF $P(\Phi_0)$

6.1. Lattice quantum field theory. Let us use quantum chromodynamics on a lattice (LQCD) as an example of a lattice QFT. LQCD is a non-perturbative approach to QCD. Calculations usually are performed using the Feynman path-integral approach. Starting point is a set of CCRs and CARs [16] for symmetric operators x_a, p_a, \tilde{x}_c , and \tilde{p}_c :

$$\begin{aligned} [q_a, p_b] &= i\delta_{a,b}, \\ [q_a, q_b] &= [p_a, p_b] = 0, \\ \{\tilde{q}_c, \tilde{p}_d\} &= i\delta_{c,d}, \\ \{\tilde{q}_c, \tilde{q}_d\} &= \{\tilde{p}_c, \tilde{p}_d\} = 0. \end{aligned}$$

The indexes represent the degrees of freedom of the quantum system, and they consist of a position, x , and a field index. In conventional QFT, x formally is continuous, but in LQCD, x is discrete and only takes finitely many values. In particular, the Feynman integrals are rigorously defined in LQCD, and calculations are performed in exactly the same way as in conventional QFT, where a space-time continuum is considered [17].

Let us assume that a and c take finitely many value, i.e. $1 \leq a \leq m$ and $1 \leq c \leq n$. Let $\mathcal{A}_{m,n}$ be the operator algebra generated by $\{x_a, p_a, \tilde{x}_c, \tilde{p}_c : 1 \leq a \leq m, 1 \leq c \leq n\}$. $\mathcal{A}_{m,n}$ is the operator algebra of the specific LQCD model instance. We define a function $x : \mathbb{C}^m \rightarrow \mathcal{A}_{m,n}$ by $x((\delta_{ab})_{1 \leq b \leq m}) = x_a$ and by complex-linear extension, and we define $x((z_a))^* = x((\bar{z}_a))$. x is a quantum field, and we analogously define the quantum fields p, \tilde{x} , and \tilde{p} . We further combine these quantum fields into one quantum field, $\Phi_{6,m,n} : V_{m,n} \rightarrow \mathcal{A}_{m,n}$, $V_{m,n} = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^n = \mathbb{C}^{2m+2n}$. Since $V_{m,n}$ is finite-dimensional, we can apply theorem 1.

Corollary 8: $\Phi_{6,m,n}$ is a quotient of Φ_0 .

In the continuum limit, the lattice spacing is supposed to approach 0, i.e. the indexes a and c take an increasing number of values ($n, m \rightarrow \infty$). We note that this does not lead to truly continuous indexes, and that continuous indexes must be differently treated. However, let us assume that for each grid size a state $\omega_{m,n}$ is determined. Since $\Phi_{6,m,n}$ is a quotient of Φ_0 , we obtain a series of states over $P(\Phi_0)$. Let us denote the series by (ω_j) for the sake of notational convenience. Since $V_{m,n} = V_{0,2m+2n}$ and since $\omega_{j,(k)} = \omega_{j,(2n+2m)}|_{P(\Phi_{0,k})}$ ($k \leq 2n+2m$), we obtain a series of states, $\omega_{j,(k)}$, on each sub-algebra $P(\Phi_{0,k})$ of $P(\Phi_0)$. If

each series, $(\omega_{j,(k)})$, converges on the corresponding sub-algebra $P(\Phi_{0,k})$, then there exists a unique limit state ω by theorem 3, and (Φ_0, ω) is a QFT by theorem 2.

Let us rephrase this result: Increasing grid sizes yield a series of states. Each state of a specific grid defines a so-called reduced state on each sub-grid. A necessary and sufficient criterion for a unique limit state in the continuum limit is that the series of reduced states converges on each sub-grid. If a limit state exists, then we obtain a QFT as defined in this paper.

6.2. String theory. We base our discussion of string theory on the lecture notes of R. J. Szabo [18]. String theory is still work in progress, and a thorough discussion of the relation of string theory to the approach in this paper is elusive so far. However, there are five different consistent formulations of string theory that are commonly seen as perturbative expansions of a unique underlying theory (M-theory), which is however not well understood yet. The five theories are related by dualities that map perturbative states in one theory to non-perturbative states in another theory.

However, the quantization of the bosonic string yields a countable set of raising and lowering operators that satisfy the relation $(a_n^\mu)^* = a_{-n}^\mu$, and, if closed strings are considered, $(\tilde{a}_n^\mu)^* = \tilde{a}_{-n}^\mu$ ($0 \leq \mu \leq d$, $n \in \mathbb{N}$). These operators satisfy CCRs. There are also zero-mode operators, x_0^μ and p_0^μ , that are conjugate to each other and that also satisfy the CCRs. The operators act on a Fock space, and we can combine them into an equivalent set of self-adjoint operators as follows:

$$\begin{aligned} x_n^\mu &= \frac{a_n^\mu + a_{-n}^\mu}{2}, & p_n^\mu &= \frac{i(a_n^\mu - a_{-n}^\mu)}{2} \\ \tilde{x}_n^\mu &= \frac{\tilde{a}_n^\mu + \tilde{a}_{-n}^\mu}{2}, & \tilde{p}_n^\mu &= \frac{i(\tilde{a}_n^\mu - \tilde{a}_{-n}^\mu)}{2}. \end{aligned}$$

We assume that observables in bosonic string theory are contained in the closure of the operator algebra, \mathcal{A}_b , generated by the operators $\{x_n^\mu, p_n^\mu, \tilde{x}_n^\mu, \tilde{p}_n^\mu\}_{n \in \mathbb{N}}$ with respect to an appropriate topology. However, let $v_n = (\delta_{n,m})_{m \in \mathbb{N}} \in V_0$ and let $x^\mu(v_{n+1}) = x_n^\mu$ and $p^\mu(v_{n+1}) = p_n^\mu$ for $n \in \mathbb{N}_0$, then, by complex-linear continuation, x and p define quantum fields over the test-function space V_0 . Analogously we can define the additional quantum fields \tilde{x} and \tilde{p} if we consider closed strings.

Fermions further need to be included in string theory to avoid inconsistencies. Canonical quantization yields another countable set of operators satisfying $(\psi_r^\mu)^* = \psi_{-r}^\mu$ ($r = 0, \pm 1, \pm \frac{1}{2}, \pm 2, \pm \frac{3}{2}, \dots$). These operators generate the corresponding operator algebra \mathcal{A}_f , and we simply denote the total operator algebra generated by operators in \mathcal{A}_b and \mathcal{A}_f by \mathcal{A} . The Hilbert spaces of the various string theories are subspaces of the full Hilbert space that is obtained by canonical quantization, and there are corresponding representations of \mathcal{A} on these sectors. As for the bosonic case, the fermionic operators can be combined into two quantum fields over V_0 . So depending on the case, we obtain four to six quantum fields over V_0 that generate the operator algebra in string theory, \mathcal{A} . These quantum fields can be combined into one quantum field, $\Phi_{7,k}$, over the test-function space $V_0^{\otimes k}$ ($k = 4$ or $k = 6$), i.e. $P(\Phi_{7,k})$ is the operator algebra in the respective string theory, and we can

apply theorem 1.

Corollary 9: Φ_0 is a core for $\Phi_{7,k}$.

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